Equation of state for dense supernova matter

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Abstract

We provide an equation of state for high density supernova matter by applying a momentum-dependent effective interaction. We focus on the study of the equation of state of high-density and high-temperature nuclear matter containing leptons (electrons and neutrinos) under the chemical equilibrium condition. Thermal effects on the properties and equation of state of nuclear matter are evaluated and analyzed in the framework of the proposed effective interaction model. Since supernova matter is characterized by a constant entropy we also present the thermodynamic properties for the isentropic case [1].

1 Introduction

Knowledge of the properties of the equation of state (EOS) of hot asymmetric nuclear matter is of fundamental importance to understand the physical mechanism of the iron core collapse of a massive star which produces a type-II supernova, and the rapid cooling of a new born hot neutron star. Additionally, the EOS defines the chemical composition, both qualitative and quantitative, of the hot nuclear matter [2, 3, 4, 5]. Supernova matter which exists in a collapsing supernova core and eventually forms a hot neutron star at birth is another form of nuclear matter distinguished in the participation of degenerate neutrinos and electrons [6]. It is characterized by almost constant entropy per baryon $S = 1 - 2$ (in units of the Boltzmann constant k_B) throughout the density n and also by a high and almost constant lepton fraction $Y_l = 0.3 - 0.4$ in contrast with ordinary neutron star matter where $S = 0$ and $Y_l \leq 0.05$. These characteristics are caused by the effects of neutrino-trapping which occurs in the dense supernova core where a neutron star is formed.

This work is a continuation of our previous work concerning the EOS of hot β-stable nuclear matter in cases where neutrinos have left the system [7]. More specifically, in order to study the properties and the EOS of hot nuclear matter, a momentum-dependent effective interaction model (MDIM) has been applied, one which is able to reproduce the results of more microscopic calculations of dense matter at zero temperature and which can be extended to finite temperature [3, 7, 8, 9]. The main incentive for the present study is the fact that only few calculations of the equation of state of the supernova matter at high densities are available, although at lower densities ($n < n_0$) (where n_0 is the saturation density) reliable results are already available.

2 The model

The model we use here, which has already been presented and analyzed in our previous papers [7, 8, 9, 10], is designed to reproduce the results of the microscopic calculations of both nuclear

and neutron-rich matter at zero temperature and can be extended to finite temperature [3, 4]. The energy density of the asymmetric nuclear matter (ANM) is given by the relation

$$\epsilon(n_n, n_p, T) = \epsilon^n_{kin}(n_n, T) + \epsilon^p_{kin}(n_p, T) + V_{int}(n_n, n_p, T), \tag{1}$$

where n_n (n_p) is the neutron (proton) density and the total baryon density is $n = n_n + n_p$. The contribution of the kinetic parts are

$$\epsilon^n_{kin}(n_n, T) + \epsilon^p_{kin}(n_p, T) = 2\int \frac{d^3k}{(2\pi)^3}\frac{\hbar^2 k^2}{2m}\left(f_n(n_n, k, T) + f_p(n_p, k, T)\right), \tag{2}$$

where f_τ, (for $\tau = n, p$) is the Fermi-Dirac distribution function.

Including the effect of finite-range forces between nucleons, the potential contribution is parameterized as follows [3]

$$\begin{aligned} V_{int}(n_n, n_p, T) &= \frac{1}{3}An_0\left[\frac{3}{2} - (\frac{1}{2} + x_0)I^2\right]u^2 + \frac{\frac{2}{3}Bn_0\left[\frac{3}{2} - (\frac{1}{2} + x_3)I^2\right]u^{\sigma+1}}{1 + \frac{2}{3}B'\left[\frac{3}{2} - (\frac{1}{2} + x_3)I^2\right]u^{\sigma-1}} \\ &+ u\sum_{i=1,2}\left[C_i\left(\mathcal{J}^i_n + \mathcal{J}^i_p\right) + \frac{(C_i - 8Z_i)}{5}I\left(\mathcal{J}^i_n - \mathcal{J}^i_p\right)\right], \end{aligned} \tag{3}$$

where

$$\mathcal{J}^i_\tau = 2\int \frac{d^3k}{(2\pi)^3}g(k, \Lambda_i)f_\tau(n_\tau, k, T). \tag{4}$$

In Eq. (3), I is the asymmetry parameter ($I = (n_n - n_p)/n$) and $u = n/n_0$, with n_0 denoting the equilibrium symmetric nuclear matter density, $n_0 = 0.16$ fm^{-3}. The asymmetry parameter I is related to the proton fraction Y_p by the equation $I = (1 - 2Y_p)$. The parameters A, B, σ, C_1, C_2 and B' which appear in the description of symmetric nuclear matter are determined in order that $E(n = n_0) - mc^2 = -16$ MeV, $n_0 = 0.16$ fm^{-3}, and the incompressibility are $K = 240$ MeV. The additional parameters x_0, x_3, Z_1, and Z_2 used to determine the properties of asymmetric nuclear matter are treated as parameters constrained by empirical knowledge [3].

The function, $g(k, \Lambda_i)$, suitably chosen to simulate finite range effects, has the following form

$$g(k, \Lambda_i) = \left[1 + \left(\frac{k}{\Lambda_i}\right)^2\right]^{-1}, \tag{5}$$

where the finite range parameters are $\Lambda_1 = 1.5k^0_F$ and $\Lambda_2 = 3k^0_F$ and k^0_F is the Fermi momentum at the saturation point n_0.

The energy density of asymmetric nuclear matter at density n and temperature T, in a good approximation, is expressed as

$$\epsilon(n, T, I) = \epsilon(n, T, I = 0) + \epsilon_{sym}(n, T, I), \tag{6}$$

where

$$\epsilon_{sym}(n, T, I) = nI^2 E^{tot}_{sym}(n, T) = nI^2\left(E^{kin}_{sym}(n, T) + E^{int}_{sym}(n, T)\right). \tag{7}$$

In Eq. (7) the nuclear symmetry energy $E^{tot}_{sym}(n, T)$ is separated into two parts corresponding to the kinetic contribution $E^{kin}_{sym}(n, T)$ and the interaction contribution $E^{int}_{sym}(n, T)$.

From Eqs. (6) and (7) and setting $I = 1$, we find that the nuclear symmetry energy $E^{tot}_{sym}(n, T)$ is given by

$$E^{tot}_{sym}(n, T) = \frac{1}{n}\left(\epsilon(n, T, I = 1) - \epsilon(n, T, I = 0)\right). \tag{8}$$

Thus, from Eq. (8) and by a suitable choice of the parameters x_0, x_3, Z_1 and Z_2, we can obtain different forms for the density dependence of the symmetry energy $E_{sym}^{tot}(n,T)$.

In a very recent work, [11] the authors carried out a systematic analysis of the nuclear symmetry energy in the formalism of the relativistic Dirac-Brueckner-Hartree-Fock approach. In this case $E_{sym}(u)$ is obtained with the simple parametrization $E_{sym}(u) = Cu^{\gamma}$ with $\gamma = 0.7 - 1.0$ and $C \approx 32$ MeV. The authors concluded that a value of γ close to 0.8 gives a reasonable description of their predictions although the use of different functions in different density regions may be best for an optimal fit [11]. The results of Ref. [11] are well reproduced by parameterizing the nuclear symmetry energy according to the following formula

$$E_{sym}^{tot}(n, T = 0) = 13u^{2/3} + 17F(u), \tag{9}$$

where the first term of the right part of Eq. (9) corresponds to the contribution of the kinetic energy and the second term to the contribution of the interaction energy.

For the function $F(u)$, which parameterizes the interaction part of the symmetry energy, we apply the following form

$$F(u) = u. \tag{10}$$

The parameters x_0, x_3, Z_1 and Z_2 are chosen so that Eq. (8), for $T = 0$ reproduces the results of Eq. (9) for the function $F(u) = u$.

2.1 Thermodynamic description of hot nuclear matter

In order to study the properties of nuclear matter at finite temperature, we need to introduce the Helmholtz free energy F which is written as [12]

$$F(n, T, I) = E(n, T, I) - TS(n, T, I). \tag{11}$$

In Eq. (11), E is the internal energy per particle, $E = \epsilon/n$, and S is the entropy per particle, $S = s/n$. From Eq. (11) it is also concluded that for $T = 0$, the free energy F and the internal energy E coincide.

The entropy density s has the same functional form as that of a non-interacting gas system, given by the equation

$$s_{\tau}(n, T, I) = -2\int \frac{d^3k}{(2\pi)^3}\left[f_{\tau}\ln f_{\tau} + (1 - f_{\tau})\ln(1 - f_{\tau})\right], \tag{12}$$

while the pressure and the chemical potentials defined as follows [12]

$$P = n^2\left(\frac{\partial(\epsilon/n)}{\partial n}\right)_{S,N_i}, \qquad \mu_i = \left(\frac{\partial \epsilon}{\partial n_i}\right)_{S,V,n_{j\neq i}}. \tag{13}$$

At this point we shall examine the properties and the EOS of nuclear matter by considering an isothermal process. In this case, the pressure and the chemical potentials are related to the derivative of the total free energy density $f = Fn$. More specifically, they are defined as follows

$$P = n^2\left(\frac{\partial(f/n)}{\partial n}\right)_{T,N_i}, \qquad \mu_i = \left(\frac{\partial f}{\partial n_i}\right)_{T,V,n_{j\neq i}}. \tag{14}$$

It is easy to demonstrate by applying Eq. (14) that (see for a proof [13] as well as [14])

$$\mu_n = F + u\left(\frac{\partial F}{\partial u}\right)_{Y_p,T} - Y_p\left(\frac{\partial F}{\partial Y_p}\right)_{n,T},$$

$$
\begin{aligned}
\mu_p &= \mu_n + \left(\frac{\partial F}{\partial Y_p}\right)_{n,T}, \\
\hat{\mu} &= \mu_n - \mu_p = -\left(\frac{\partial F}{\partial Y_p}\right)_{n,T}.
\end{aligned} \tag{15}
$$

We can define the symmetry free energy per particle $F_{sym}(n,T)$ by the following parabolic approximation (see also [14, 15])

$$
\begin{aligned}
F(n,T,I) &= F(n,T,I=0) + I^2 F_{sym}(n,T) \\
&= F(n,T,I=0) + (1-2Y_p)^2 F_{sym}(n,T),
\end{aligned} \tag{16}
$$

where

$$
F_{sym}(n,T) = F(n,T,I=1) - F(n,T,I=0). \tag{17}
$$

It is worth noting that the above approximation is not valid from the beginning, but one needs to check the validity of the parabolic law in the present model before using it. In Ref. [7] we have proved the validity of the approximation (16).

Now, by applying Eq. (16) in Eq. (15), we obtain the key relation

$$
\hat{\mu} = \mu_n - \mu_p = 4(1-2Y_p)F_{sym}(n,T). \tag{18}
$$

The above equation is similar to that obtained for cold nuclear matter by replacing $E_{sym}(n)$ with $F_{sym}(n,T)$.

2.2 β-equilibrium in hot proto-neutron star and supernova

Stable high density nuclear matter must be in chemical equilibrium with all type of reactions, including the weak interactions in which β decay and electron capture take place simultaneously

$$
n \longrightarrow p + e^- + \bar{\nu}_e, \qquad\qquad p + e^- \longrightarrow n + \nu_e. \tag{19}
$$

Both types of reactions change the electron per nucleon fraction, Y_e and thus affect the equation of state. In a previous study, we assumed that neutrinos generated in these reactions left the system [7]. The absence of neutrino-trapping has a dramatic effect on the equation of state and is the main cause of a significant reduction in the values of the proton fraction Y_p [6, 16]. The equation of state of hot nuclear matter in β-equilibrium (considering that it consists of neutrons, protons, electrons and neutrinos) can be obtained by calculating the total energy density ϵ_{tot} as well as the total pressure P_{tot}. The total energy density is given by

$$
\epsilon_{tot}(n,T,I) = \epsilon_b(n,T,I) + \sum_{l=e,\nu_e} \epsilon_l(n,T,I), \tag{20}
$$

where $\epsilon_b(n,T,I)$ and $\epsilon_l(n,T,I)$ are the contributions of baryons and leptons respectively. The total pressure is

$$
P_{tot}(n,T,I) = P_b(n,T,I) + \sum_{l=e,\nu_e} P_l(n,T,I), \tag{21}
$$

where $P_b(n,T,I)$ is the contribution of the baryons i.e.

$$
P_b(n,T,I) = T\sum_{\tau=p,n} s_\tau(n,T,I) + \sum_{\tau=n,p} n_\tau \mu_\tau(n,T,I) - \epsilon_b(n,T,I), \tag{22}
$$

while $P_l(n,T,I)$ is the contribution of the leptons. From Eqs. (20) and (21) we can construct the isothermal curves for energy and pressure and finally derive the isothermal behavior of the equation of state of hot nuclear matter under β-equilibrium.

3 Results and Discussions

We calculate the equation of state of hot asymmetric nuclear matter by applying a momentum dependent effective interaction model describing the baryons interaction. We consider that nuclear matter contains neutrons, protons, electrons and neutrinos under β-equilibrium and charge neutrality. The key quantities in our calculations are the proton fraction Y_p and also the asymmetry free energy defined in Eq. (17). It is worth pointing out that since the supernova explosion itself is a dynamic phenomenon, the chemical composition of matter changes according to the evolution of the star all the time [17]. During supernova explosion, the chemical composition of matter reaches equilibrium not in the whole star but locally. In our present work we assume matter in the chemical equilibrium for simplicity in order to analyze the properties of hot neutron star and supernova matter.

Following the discussion of Takatsuka et al. [6], we attempt to extend the discussion concerning the dependence of equilibrium fraction $Y_e(= Y_p)$ on the baryon density as well as on the nuclear symmetry energy. We ignore the temperature effect to clarify the situation. Actually, the situation does not change by including finite temperature effects. The energy per baryon of supernova matter E_{SM} and cold neutron star matter E_{NS} are expressed as function of n and Y_p (see also ref. [6]) as

$$\begin{aligned}
E_{SM}(n, Y_p) &= E_b(n, Y_p) + E_e(n, Y_p) + E_{\nu_e}(n, Y_p) \qquad (23)\\
&= E_b(n, Y_p = 0.5) + E_{sym}(n)(1 - 2Y_p)^2\\
&+ 253.6u^{1/3}Y_p^{4/3} + 319.516u^{1/3}(Y_l - Y_p)^{4/3},\\
E_{NS}(n, Y_p) &= E_b(n, Y_p) + E_e(n, Y_p) \qquad (24)\\
&= E_b(n, Y_p = 0.5) + E_{sym}(n)(1 - 2Y_p)^2 + 253.6u^{1/3}Y_p^{4/3}.
\end{aligned}$$

$E_{sym}(n)$ is plotted in Fig. 1(a) for the three different parametrizations. In the same figure we have included recent results provided in reference [11] achieved by performing microscopic calculations in asymmetric nuclear matter. In this case $E_{sym}(n)$ is obtained with the simple parametrization

$$E_{sym}(u) = Cu^{\gamma},$$

with $\gamma = 0.8$ and $C = 32$ MeV. It is obvious that the results of the above parametrization, correspond very well with the parametrization $F(u) = u$ which is proposed here.

The equilibrium proton fraction Y_p is calculated by solving the equation $\partial E_{SM,NS}/\partial Y_p = 0$ for various values of the density n, $E_{sym}(n)$ and $Y_l = 0.4$ for supernova matter (see Fig. 1(b)). In the case of cold neutron star matter, Y_p depends strongly on both the baryon density and the values of the $E_{sym}(n)$. This is not the case for supernova matter where the effect of nuclear symmetry energy in determining Y_p is less important than in cold neutron star matter. In addition, Y_p, for a fixed parametrization of $F(u)$ is almost constant with respect to n.

In Fig. 2 we plot the contribution of the baryons S_b, leptons S_l and the total S_{tot} to the entropy per baryon. In all cases, S is a decreasing function of the baryon density n. Temperature affects appreciably both baryon and lepton contribution. It should be noted that the contribution of baryons S_b may be written as $S_b = S_{kin} + S_{int}$, where the term S_{kin} originates from the temperature effect on the kinetic part of the energy density and S_{int} reflects thermal effects on the potential energy density.

In Fig. 3, we display the contribution to internal energy E from baryons E_b and leptons E_l for $Y_l = 0.3$ and $Y_l = 0.4$ and for various values of T. The most striking aspect is that the lepton energy, $E_l = E_e + E_{\nu_e}$, dominates in the internal energy of the matter up to $n \sim 0.6$ fm^{-3} (for $Y_l = 0.3$) and $n \sim 0.8$ fm^{-3} (for $Y_l = 0.4$). This is a characteristic of the supernova matter and is in remarkable contrast with the situation of cold neutron star matter [6]. The contribution from baryon E_b gets larger with the increase of n and is comparable with E_l for high values of n.

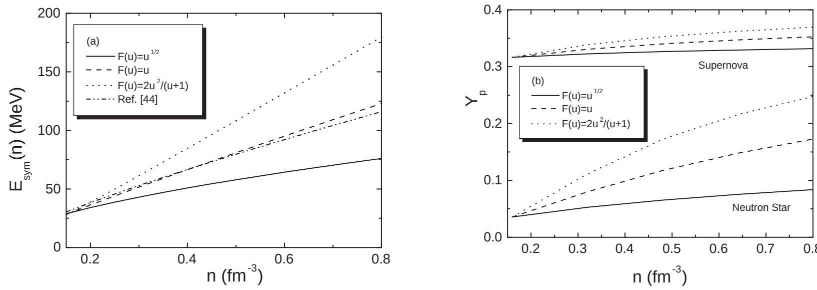

Figure 1: a) The nuclear symmetry energy for three different parametrization of the interaction part with the results of reference [11] (see text for more details) and b) the proton fraction Y_p versus baryon density for cold neutron star matter (down curves) and supernova matter (up curves) for the three different parametrization of the nuclear symmetry energy.

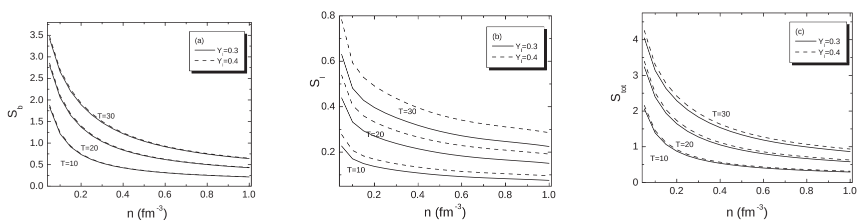

Figure 2: Contribution to the total entropy per particle of a) baryons (S_b), b) leptons (S_l) and c) the total entropy (S_{tot}) versus the baryon density for various values of T for total lepton fraction $Y_l = 0.3$ and $Y_l = 0.4$.

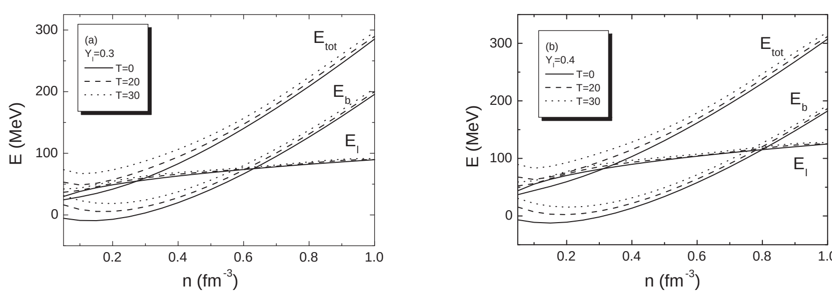

Figure 3: Contribution to the total energy per particle of baryons (E_b), leptons (E_l) and the total energy (E_{tot}) versus the baryon density for various values of T for total lepton fraction a) $Y_l = 0.3$ and b) $Y_l = 0.4$.

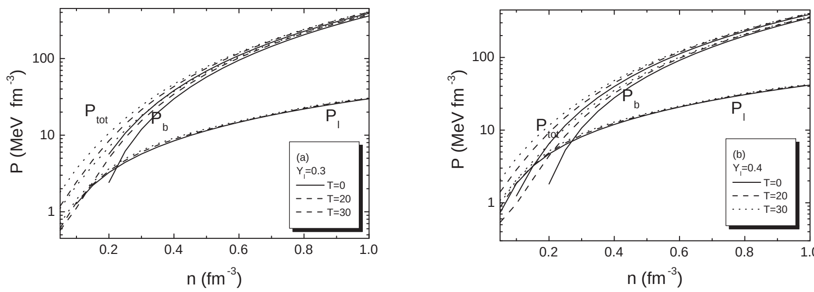

Figure 4: The pressure of baryons (P_b), leptons (P_l) and the total pressure (P_{tot}) versus the baryon density for various values of T for the cases a) $Y_l = 0.3$ and b) $Y_l = 0.4$.

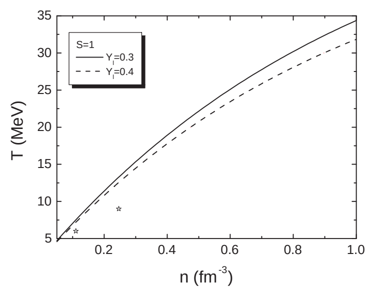

Figure 5: Temperature T-density n relation with $Y_l = 0.3$ (solid line) and $Y_l = 0.4$ (dashed line) for $S = 1$. Stars denote the results for the case with $S = 1$ and $Y_l = 0.4$, extracted from the results by Lattimer et al. [19].

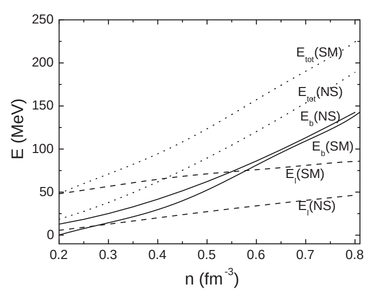

Figure 6: Internal energy per baryon versus density n for dense supernova matter (SM) in comparison with that of cold neutron star matter (NS) by applying the same model. The case of supernova matter corresponds to $S = 1$ and $Y_l = 0.3$. The contribution of each species is plotted separately.

The contributions of baryon and leptons on the total pressure are presented in Fig. 4. In contrast to the situation of the internal energy, the nuclear part contribution plays a more important role compared with the lepton part. The lepton pressure P_l is comparable to baryon pressure P_b up to $n \sim 0.2$ fm^{-3}, but for higher values of n it is significantly small.

As pointed out by Bethe et al. [18], the crucial feature in determining the evaluation of a collapsing pre-supernova core is that the entropy per particle is very low, of the order of unity (in units of the Boltzmann constant k_B), and nearly constant during all the stages of the collapse up to the shock wave formation. Therefore, the collapse is an adiabatic process of a highly ordered system. So, since the supernova matter is characterized by a constant entropy and constant lepton fraction, we shall also discuss the properties under this condition. This can be done by converting the results for isothermal case (T=const) into those for adiabatic case (S=const) in terms of the $T - n$ relation constrained by a constant entropy.

The $T = T(n)$ relation is constructed by $\{T, n\}$ values to satisfy $S(n, T)$=const in an $S - n$ diagram. Fig. 5 shows the results for $Y_l = 0.3$ and $Y_l = 0.4$ for $S = 1$. Temperature is an increasing function of n. Furthermore, for the same density, the temperature is higher for lower values of Y_l. The values of T for various values of n are derived, for the two cases, with the least-squares fit method and found to take the general form

$$T(n) = an^b,$$

where $a = 35.412$, $b = 0.70481$ for $Y_l = 0.3$ and $a = 32.35706$, $b = 0.67694$ for $Y_l = 0.4$. The results of this study are in very good agreement with those of Takatsuka et al. [6]. The stars at lower density denote the $\{T, n\}$ values for $S = 1$ and $Y_l = 0.4$ which are derived from Lattimer et al. [19]. It is concluded that the temperature increases considerably when moving from the outer part of the star to the center in order to maintain a constant value of the entropy per baryon.

Finally, in Fig. 6 we compare the EOS's between supernova matter and cold neutron star matter. The case for supernova matter corresponds to $S = 1$ and $Y_l = 0.3$. It is thus clear that the internal energy E_{tot} of supernova matter (SM) is remarkably larger than that of neutron star matter (NS). As far as the nucleon part E_b is concerned, the E_b in SM is slightly lower than that in NS due to the large energy gain in symmetry energy (see also [6]). However, the lepton contribution on the internal energy E_l is remarkably larger in SN matter compared to NS matter due to the effect of a large lepton fraction, that is, a large kinetic energy of abundant leptons. High temperature also contributes to the stiffening, but it is less effective than the high lepton fractions.

4 Summary

The evaluation of the equation of state of hot nuclear matter is a major challenge for nuclear physics and astrophysics. EOS is the basic ingredient necessary for studying the supernova explosion as well as for determining the properties of hot neutron stars. The motive for the present work has been to apply a momentum-dependent interaction model for the study of the hot nuclear matter EOS under β-equilibrium. Special attention has been dedicated to the study of the contribution of the components of β-stable nuclear matter on the entropy per particle, a quantity of great interest in the study of structure and collapse of supernova. The above EOS can be applied to the evaluation of the gross properties of hot neutron stars i.e. mass and radius.

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